Related Rates - Worked Examples

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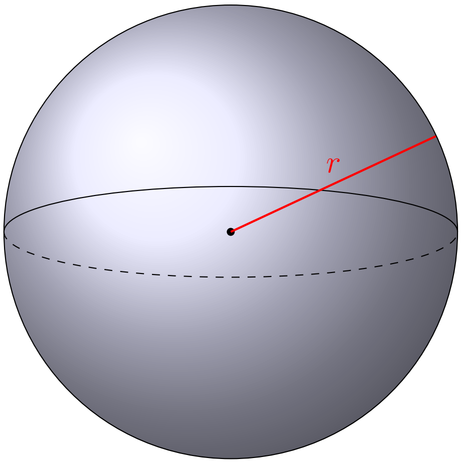
Expanding Sphere

Question

Consider a spherical balloon whose **volume** is expanding at a **rate of 0.5 cm³/sec**. How fast is the **radius changing** when the **radius** is **10 cm**?

Diagram



Solution

Let V = volume of the balloon, r = radius of the balloon, and t = time.

We know $\frac{dV}{dt} = 0.5$ cm^3/sec, and we want to find $\frac{dr}{dt}$ when $r = 10$ cm. Using the chain rule, we can write:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

Also, note that the volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

Using this equation for the volume, we can differentiate to find $\frac{dV}{dr}$.

$$\begin{aligned}\frac{dV}{dr} &= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \\ &= \frac{4}{3}\pi\left(3r^2\right) \\ &= 4\pi r^2\end{aligned}$$

Plugging this, as well as $\frac{dV}{dt}$, into our chain rule equation gives:

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ 0.5 &= 4\pi r^2 \cdot \frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{0.5}{4\pi r^2}\end{aligned}$$

Now, substitute $r = 10$ into this expression.

$$\begin{aligned}\left.\frac{dr}{dt}\right|_{r=10} &= \frac{0.5}{4\pi\,(10)^2}\\ &= \frac{0.5}{4\pi\cdot 100}\\ &= \frac{0.5}{400\pi}\\ &= \frac{1}{800\pi}\ \text{cm/sec}\\ &\approx 0.000398\ \text{cm/sec}\\ &\approx 3.98\times 10^{-4}\ \text{cm/sec}\end{aligned}$$

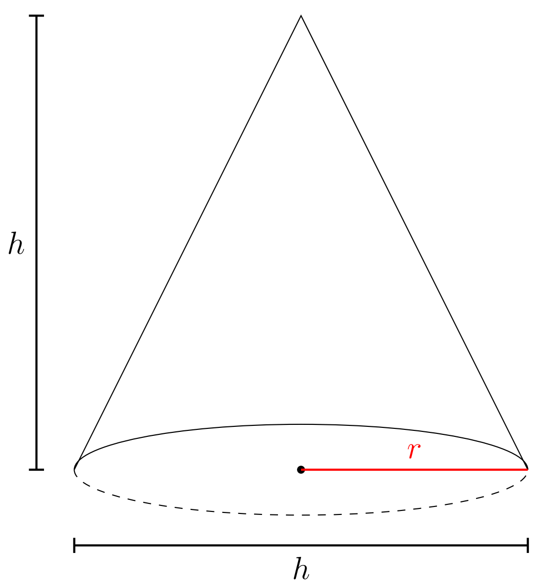
Therefore, the radius of the balloon is changing at a rate of approximately 3.98×10^{-4} cm/sec when the radius is 10 cm.

Gravel Pile

Question

Gravel falls off of a conveyor belt at a **rate of 5 m^3/sec**. This forms a cone-shaped pile whose base's **diameter is equal to its height**. How quickly is the **height of the pile changing** when the pile is **15 m high**?

Diagram



Solution

Let h = height of the pile, r = radius of the pile, and t = time.

We know $\frac{dV}{dt} = 5$ m^3/sec, and we want to find $\frac{dh}{dt}$ when $h = 15$. Using the chain rule, we can write:

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

The volume of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

Since we know the base's diameter and the height are equal (i.e., $h = 2 \cdot r$), we can rewrite the volume as:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12}h^3$$

Using this equation for the volume, we can differentiate to find $\frac{dV}{dh}$.

$$\begin{aligned}\frac{dV}{dh} &= \frac{d}{dh}\left(\frac{\pi}{12}h^3\right) \\ &= \frac{\pi}{12}\left(3h^2\right) \\ &= \frac{\pi}{3}h^2\end{aligned}$$

Plugging this, as well as $\frac{dV}{dt}$, into the chain rule equation gives:

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt} \\ 5 &= \frac{\pi}{3}h^2 \cdot \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{15}{\pi h^2}\end{aligned}$$

Now, substitute $h = 15$ into this expression.

$$\begin{aligned}\left.\frac{dh}{dt}\right|_{h=15} &= \frac{15}{\pi\left(15\right)^2} \\ &= \frac{1}{15\pi} \text{ m/sec} \\ &\approx 0.0212206591 \text{ m/sec} \\ &\approx 2.12 \times 10^{-2} \text{ m/sec}\end{aligned}$$

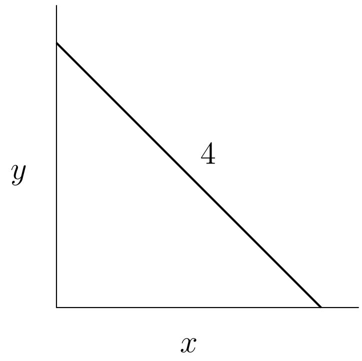
Therefore, the height of the pile is changing at a rate of approximately 2.12×10^{-2} m/sec when the height is 15 m.

Falling Ladder

Question

Consider a **4 meter long ladder** leaning against a wall. The **bottom** of the ladder **slides away from the wall at a rate of 1 m/sec**. How fast is the **top of the ladder sliding down the wall** when the **bottom is 3 m away** from the wall?

Diagram



Solution

Let x = horizontal distance from the wall, and y = vertical height on the wall.

We know $\frac{dx}{dt} = 1$ m/sec, and we want to find $\frac{dy}{dt}$ when $x = 3$. Using the chain rule, we can write:

$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$$

Since the ladder forms a right triangle, we can relate x and y by the Pythagorean Theorem:

$$x^2 + y^2 = 4^2 \implies x^2 + y^2 = 16$$

In order to find $\frac{dx}{dy}$, we can differentiate the above equation with respect to y.

$$\begin{aligned}
\frac{d}{dy}\left(x^2 + y^2\right) &= \frac{d}{dy}(16) \\
2x\frac{dx}{dy} + 2y &= 0 \\
\frac{dx}{dy} &= -\frac{2y}{2x} \\
\frac{dx}{dy} &= -\frac{y}{x}
\end{aligned}$$

Plugging this, as well as $\frac{dx}{dt}$, into the chain rule equation gives:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{dy} \cdot \frac{dy}{dt} \\
1 &= -\frac{y}{x} \cdot \frac{dy}{dt} \\
\frac{dy}{dt} &= -\frac{x}{y}
\end{aligned}$$

We need to substitute $x = 3$ into this expression, but we also need to figure out the corresponding y value to also plug in. To do this, we can use the Pythagorean Theorem.

$$\begin{aligned} x^2 + y^2 &= 4^2 \\ 3^2 + y^2 &= 16 \\ 9 + y^2 &= 16 \\ y^2 &= 7 \\ y &= \pm\sqrt{7} \\ y &= \sqrt{7} \qquad \text{Ignore the negative solution} \end{aligned}$$

Now, substitute both $x = 3$ and $y = \sqrt{7}$ into the expression for $\frac{dy}{dt}$.

$$\begin{aligned} \left.\frac{dy}{dt}\right|_{x=3, y=\sqrt{7}} &= -\frac{3}{\sqrt{7}} \text{ m/sec} \\ &\approx -1.133893419 \text{ m/sec} \end{aligned}$$

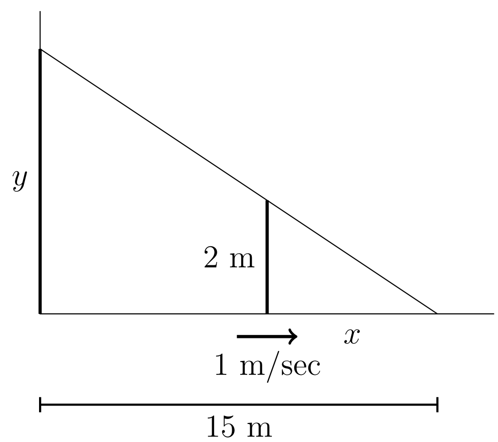
Therefore, the top of the ladder is falling down the wall at a rate of approximately 1.13 m/sec when the bottom of the ladder is 3 m from the wall.

Moving Shadow

Question

Consider a **2 m tall** person walking towards a light that is **15 m** from a wall at a **rate of 1 m/sec**. How **quickly is the shadow of the person moving up the wall** when they are **5 m** from the light?

Diagram



Solution

Let $x =$ distance between the person and the light, and $y =$ height of the shadow on the wall.

We know $\frac{dx}{dt} = -1$ m/sec, and we want to find $\frac{dy}{dt}$ when $x = 5$. Using the chain rule, we can write:

$$\frac{dy}{dt} = \frac{dx}{dt} \cdot \frac{dy}{dx}$$

Note that the height of the shadow and distance between the light and wall form a similar triangle with the height of the person, and the distance to the light. Thus,

$$\frac{y}{15} = \frac{2}{x} \implies y = \frac{30}{x}$$

Differentiating gives us:

$$\begin{aligned} \frac{dy}{dt} &= -\frac{30}{x^2} \cdot \frac{dx}{dt} \\ &= -\frac{30}{x^2} \cdot -1 \\ &= \frac{30}{x^2} \end{aligned}$$

Now, we can substitute $x = 5$ into the expression to get our final answer.

$$\begin{aligned} \left.\frac{dy}{dt}\right|_{x=5} &= \frac{30}{5^2} \\ &= \frac{30}{25} \text{ m/sec} \\ &= 1.2 \text{ m/sec} \end{aligned}$$

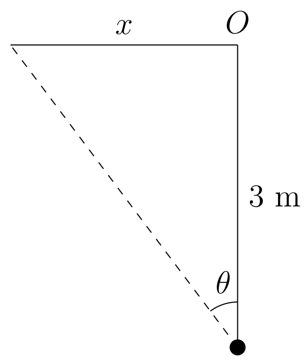
Therefore, the shadow is moving up the wall at a rate of 1.2 m/sec when the person is 5 m from the light.

Rotating Light

Question

Consider a light rotating at a **rate of 2 rad/sec**, shining on a wall **3 meters away**. How **quickly is the light moving along the wall** when the light is **5 meters** away from the perpendicular point (O) on the wall?

Diagram



Solution

Let $\theta =$ the angle between the light and the perpendicular from the light to the wall, and $x =$ the horizontal distance between the perpendicular point and the light.

We know $\frac{d\theta}{dt} = 2$ rad/sec, and we want to find $\frac{dx}{dt}$ when $x = 5$. Using the chain rule, we can write:

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$$

By using right triangle trigonometry, we can find an expression for θ.

$$\begin{aligned}
\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\
\tan\theta &= \frac{x}{3} \\
\arctan(\tan\theta) &= \arctan\left(\frac{x}{3}\right) \\
\theta &= \arctan\left(\frac{x}{3}\right)
\end{aligned}$$

Now, differentiate and rearrange to find the derivative we are looking for.

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} \cdot \frac{dx}{dt} \\
2 &= \frac{1}{1+\left(\frac{x}{3}\right)^2} \cdot \frac{1}{3} \cdot \frac{dx}{dt} \\
\frac{dx}{dt} &= 6\left(1+\left(\frac{x}{3}\right)^2\right) \\
&= 6\left(1+\frac{x^2}{9}\right) \\
&= 6+\frac{2}{3}x^2
\end{aligned}$$

Finally, substitute $x = 5$ into the expression for the derivative.

$$\begin{aligned}
\left.\frac{dx}{dt}\right|_{x=5} &= 6+\frac{2}{3}(5)^2 \\
&= 6+\frac{2}{3}\cdot 25 \\
&= \frac{18}{3}+\frac{50}{3} \\
&= \frac{68}{3} \text{ m/sec} \\
&\approx 22.\overline{66} \text{ m/sec}
\end{aligned}$$

Therefore, the light is moving at a rate of approximately 22.7 m/sec when the light is 5 meters from the perpendicular point on the wall.